On SDP formulations for quadratic optimization with indicator variables

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Quadratic optimization with indicator variables

$$\begin{aligned} \min_{x,y}\ & a'x + b'y + y'Qy \\ \text{s.t. }\ & y_i(1-x_i) = 0, \quad \forall i \in [n] \\ & x \in \{0,1\}^n,\ y \in \mathbb{R}^n_+, \end{aligned} \tag{MIO}$$

where $x_i = \mathbb{I}_{\{y_i \neq 0\}}$.

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- Portfolio optimization (Bienstock 1996)
- Optimal control (Gao and Li 2011)
- Signal denoising (Bach 2016)
- ...

Mixed-integer optimization

The problem can be naturally model using big-M:

$$\min_{x,y} \; a'x + b'y + y'Qy$$
$$\text{s.t.} \;\; y_i(1 - x_i) = 0, \quad \forall i \in [n] \qquad \text{(MIO)}$$
$$x \in \{0,1\}^n, \; y \in \mathbb{R}_+^n$$

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Highly depends on the selection of M and poor relaxation quality.

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Highly depends on the selection of M and poor relaxation quality.

Approach: construct strong convex relaxations of (MIO).

Perspective reformulation

When Q is PSD and diagonal, the problem is separable.

$$\begin{aligned} \min_{x,y} \ & \sum_{i\in[n]} a_i x_i + b_i y_i + Q_{ii} y_i^2 \\ \text{s.t.} \ & y_i(1-x_i) = 0, \quad \forall i \in [n] \\ & x \in \{0,1\}^n, \ y \in \mathbb{R}_+^n, \end{aligned}$$

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$$\min_{x,y} \sum_{i\in[n]} a_i x_i + b_i y_i + Q_{ii} y_i^2/x_i$$
$$\text{s.t. } x \in [0,1]^n,\ y \in \mathbb{R}_+^n,$$

where $0/0 = 0$ and $a/0 = +\infty$ when $a \neq 0$.

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Ideal formulation!

Perspective reformulation

When (MIO) is not separable, introduce $Y \approx yy'$

$$\begin{aligned} \min\ & a'x + b'y + \langle Q, Y \rangle \\ \text{s.t.}\ & Y \succeq yy' \\ & y_i^2 \le Y_{ii} x_i \quad \forall i \in [n] \\ & 0 \le x \le 1 \\ & y \ge 0, \end{aligned} \tag{Persp}$$

where $Y \in \mathbb{R}^{n \times n}$.

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where $Y \in \mathbb{R}^{n \times n}$.

However, when Q deviates from diagonal, the performance deteriorates.

Standard semidefinite programming reformulation

Introduce $Z \approx \begin{pmatrix} y \\ x \end{pmatrix} \begin{pmatrix} y' & x' \end{pmatrix}$,

$$\begin{aligned} \min\ & a'x + b'y + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} Z_{ij} \\ \text{s.t. } & y_i - Z_{i,i+n} = 0 && \forall i \in [n] \\ & x_i - Z_{i+n,i+n} = 0 && \forall i \in [n] \\ & Z - \begin{pmatrix} y \\ x \end{pmatrix} \begin{pmatrix} y' & x' \end{pmatrix} \succeq 0 \\ & 0 \le x \le 1. \end{aligned} \qquad (\mathrm{SDP_s})$$

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\end{aligned}
\tag{SDP$_\mathrm{s}$}
$$

- max-cut problem (Goemans and Williamson 1995)
- matrix completion (Candes and Plan 2010)
- ...

PR vs. Standard SDP

Which one is stronger among PR and Standard SDP?

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PR:

$$\min\ a'x + b'y + \langle Q, Y\rangle$$
$$\text{s.t.}\ \begin{pmatrix} 1 & y_1 & y_2 \\ y_1 & Y_{11} & Y_{12} \\ y_2 & Y_{12} & Y_{22} \end{pmatrix} \succeq 0$$
$$y_1^2 \le Y_{11}x_1,\ y_2^2 \le Y_{22}x_2$$
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SPD_s:

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$$\mathrm{PR} \Longrightarrow \mathrm{SDP_s}$$

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Proposition

(PR) is equivalent to (SDP_s).

SDP Strengthening

Quadratic constraint with two indicators:

$$X_+ = \left\{(x, y, t) \in \{0, 1\}^2 \times \mathbb{R}^3_+ : t \geq d_1 y_1^2 + 2y_1 y_2 + d_2 y_2^2, (1 - x) \circ y = 0\right\},$$

where $d_1 > 0, d_2 > 0, d_1 d_2 \geq 1$.

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$$\begin{aligned} f_+^*(x, y; d) := \min_{z,\lambda} & \frac{d_1}{x_1 - \lambda}(y_1 - z_1)^2 + \frac{d_2}{x_2 - \lambda}(y_2 - z_2)^2 + \frac{d_1 z_1^2 + 2z_1 z_2 + d_2 z_2^2}{\lambda} \\ \text{s.t.} \quad & z_1 \geq 0, z_2 \geq 0 \\ & \max\{0, x_1 + x_2 - 1\} \leq \lambda \leq \min\{x_1, x_2\}. \end{aligned}$$

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$$\text{conv}(X_+) = \left\{(x,y,t) \in [0,1]^2 \times \mathbb{R}^3_+ : t \geq f_+^*(x,y;d)\right\}.$$

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We also provide a description of $f_+^*(\cdot)$ in the original space of variables.

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Inequality $t \geq f^*_+(\cdot)$ is SOCP-representable in lifted space

Convex relaxation via 2×2 decomposition

When Q is 2×2 decomposable,

$$y'Qy = \sum_{i \in [n]} D_{ii} y_i^2 + \sum_{i \neq j} c_{ij} (d_1^{ij} y_i^2 + 2 y_i y_j + d_2^{ij} y_j^2),$$

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$$\Rightarrow \min \; a'x + b'y + \sum_{i=1}^{n} D_{ii} \frac{y_i^2}{x_i} +$$

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$$\Rightarrow \min \;\; a'x + b'y + \sum_{i=1}^{n} D_{ii} \frac{y_i^2}{x_i} + \sum_{i \neq j} c_{ij} f_+^*(x_{i,j}, y_{i,j}; d^{ij}),$$

where $x_{i,j} = (x_i, x_j)$ and $y_{i,j} = (y_i, y_j)$

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Issue:

- There are potentially infinite ways to decompose Q!

Convex relaxation via 2×2 decomposition

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where $x_{i,j} = (x_i, x_j)$ and $y_{i,j} = (y_i, y_j)$

Issue:

- There are potentially infinite ways to decompose Q!
- What if Q is not 2×2 decomposable?

Build new valid inequalities

In general, introduce $Y \approx yy'$

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Valid for all $d^{ij} > 0$ such that $d_1^{ij} d_2^{ij} \geq 1$! $\rightarrow$

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Valid for all $d^{ij} > 0$ such that $d_1^{ij} d_2^{ij} \geq 1$! $\rightarrow$ Take max

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Proposition

Above formulation is a valid convex relaxation of (MIO).

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cutting surface?

Valid inequality implementation

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The following formulation is a valid convex relaxation of (MIO) and is stronger than Persp/SDP$_s$.

$$
\begin{aligned}
\min\ & a'x + b'y + \langle Q, Y \rangle \\
s.t.\ & Y - yy' \succeq 0 \\
& W^{(ij)} \succeq 0 && \forall i > j \\
& W^{(ij)}_{12} = Y_{ij} && \forall i > j \\
& (Y_{ii} - W^{(ij)}_{11})(x_i - W^{(ij)}_{33}) \geq (y_i - W^{(ij)}_{31})^2, W^{(ij)}_{11} \leq Y_{ii}, W^{(ij)}_{33} \leq x_i && \forall i > j \\
& (Y_{jj} - W^{(ij)}_{22})(x_j - W^{(ij)}_{33}) \geq (y_j - W^{(ij)}_{32})^2, W^{(ij)}_{22} \leq Y_{jj}, W^{(ij)}_{33} \leq x_j && \forall i > j \\
& 0 \leq W^{(ij)}_{31} \leq y_i, 0 \leq W^{(ij)}_{32} \leq y_j && \forall i > j \\
& W^{(ij)}_{33} \geq x_i + x_j - 1 && \forall i > j \\
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\end{aligned}
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Example when $n = 2$ (continued)

Example

For $n = 2$, the instance of (MIO) with

$$a = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, b = \begin{pmatrix} -8 \\ -5 \end{pmatrix}, Q = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Optimal solution

	obj val	x_1	x_2	y_1	y_2
Persp	-2.866	0.049	0.268	0.208	1.369
SDP_s	-2.866	0.049	0.268	0.208	1.369

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$$a = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, b = \begin{pmatrix} -8 \\ -5 \end{pmatrix}, Q = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Optimal solution

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SDP_s	-2.866	0.049	0.268	0.208	1.369
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SDP_p delivers the optimal solution!

Computational experiment

Consider portfolio index tracking problem of the form

$$\begin{aligned}\min_{x,y}\ & (y-y_B)'Q(y-y_B)\\ \text{s.t.}\ & 1'y=1, 1'x\le k\\ & 0\le y\le x, x\in\{0,1\}^n\end{aligned}$$

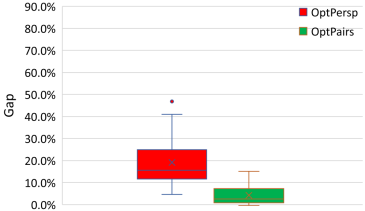
- $y_B \in \mathbb{R}^n$ is a benchmark index portfolio
- Q is the covariance matrix of security returns
- k is the maximum number of securities in the portfolio

Computational results

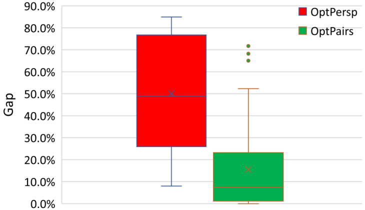
Distribution of gaps for OptPersp and OptPairs

OptPersp: application of Persp

OptPairs: application of our new relaxation



(A) Data since 2010.
OptPersp 19.1% v.s. OptPairs 4.2%



(B) Data since 2015.
OptPersp 50.1% v.s. OptPairs 15.5%

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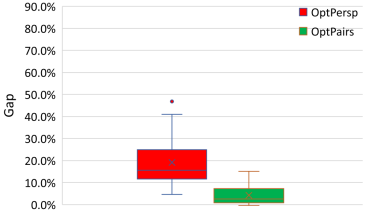

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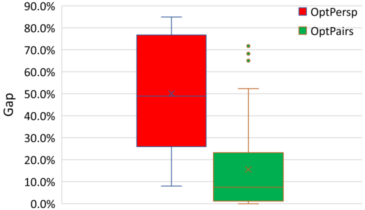

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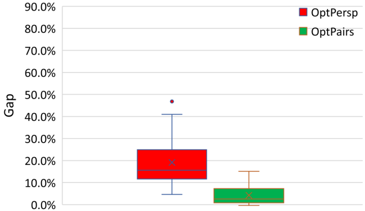

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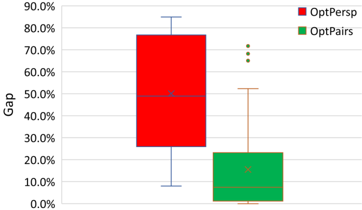

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Computational results

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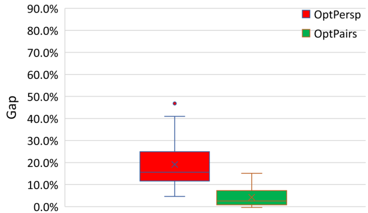

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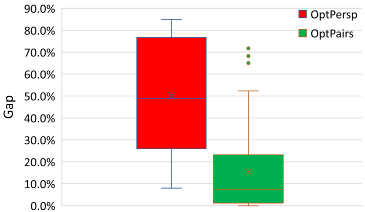

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Our paper is available at:

www.optimization-online.org/DB_HTML/2020/04/7746.html

Thank You!

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