Fractional 0-1 programming and Submodularity

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Collaborators

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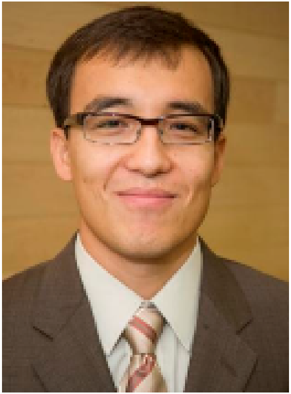
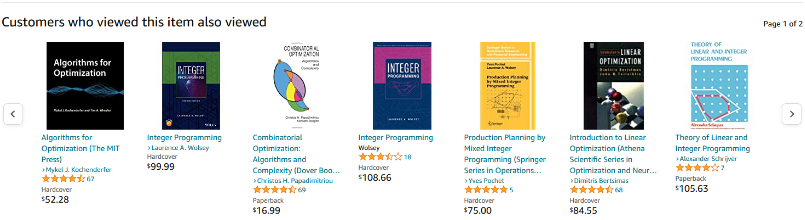


Oleg Prokopyev
University of Pittsburgh

One motivating example

Assortment optimization

Search result of “Integer programming”



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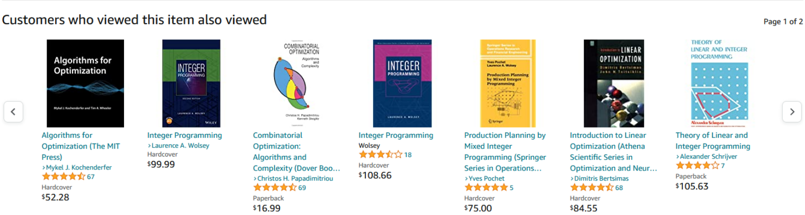

How does a company decide which products to display?

Motivating example: assortment optimization problem

Goal: choose an assortment of products to maximize profits under the MMNL model

- $[n]$: set of products offered to customers
- $[m]$: set of market segments
- v: preference weights
- r: revenue rates
- x: $x_i = 1$ iff $i \in S$

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Introduction

Multiple-ratio fractional 0-1 program

$$\max_{x \in \mathcal{F}} \sum_{k \in [m]} \frac{\sum_{i \in [n]} a_{ki} x_i}{b_{k0} + \sum_{i \in [n]} b_{ki} x_i} \tag{1}$$

where $a > 0, b > 0$ and $\mathcal{F} \subseteq \{0,1\}^n$ is the feasible region

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- Minimum fractional spanning tree problem (Ursulenko et al. 2013)
- ...

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$$x = \mathbb{I}_S := \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{o.w.} \end{cases} \leftrightarrow S = \{i : x_i = 1\}$$

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- Capacity constraint: $\mathcal{F} = \{S \subseteq [n] : \sum_{i \in S} w_i \leq c\}$, $w > 0, c > 0$

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How hard is it?

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Very challenging!

Submodularity

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where $g(t) = t/(b_0 + t)$ is concave; see, e.g. Benati and Hansen (2002)

Submodularity characterization of a single ratio

In general:

Theorem (Han et al. (2020))

Function $h(\cdot)$ is submodular over $\mathcal{F}$ if and only if

$$h(S \cup \{i\}) + h(S \cup \{j\}) \leq \frac{a_i}{b_i} + \frac{a_j}{b_j}$$

for all $S \subseteq N$, and $i, j \notin S$ with $i \neq j$ such that $S \cup \{i\} \cup \{j\} \in \mathcal{F}$.

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In the context of assortment optimization:

Proposition (Han et al. (2020))

If

$$\frac{r_{\max} - r_{\min}}{r_{\max}} \leq \min_{S \in \mathcal{F}} \frac{\mathbb{P}\{\text{customer leave with no purchase}; S\}}{\mathbb{P}\{\text{customer make a purchase}; S\}},$$

where $r_{\max} = \max_i a_i/b_i$, $r_{\min} = \min_i a_i/b_i$, then $h(x)$ is submodular.

Monotonicity and submodularity

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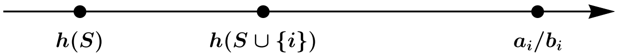

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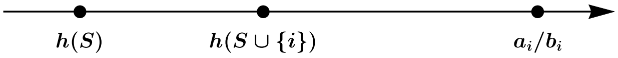
$$h(S) \le h(S \cup \{i\}) \Leftrightarrow h(S \cup \{i\}) \le a_i/b_i$$

$$\Rightarrow \quad h(S \cup \{i\}) + h(S \cup \{j\}) \le \frac{a_i}{b_i} + \frac{a_j}{b_j}.$$



Monotonicity and submodularity

Does submodularity $\Rightarrow$ monotonicity?

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Does submodularity ⇒ monotonicity?

Example: Consider

$$h(x) = \frac{3x_1 + 2x_2 + x_3}{2 + x_1 + x_2 + x_3},$$

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$$h(\{3\}) = \frac{1}{3} < h(\{1,2,3\}) = \frac{6}{5} < h(\{1,2\}) = \frac{5}{4}$$

$\Rightarrow$ monotonicity fails to hold.

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Proposition (Han et al. (2020))

If $h(x)$ is submodular over $\mathcal{F}$, then it is monotone nondecreasing over $\mathcal{F}_1 := \{S \in \mathcal{F} : n \in S\}$ and $\mathcal{F}_2 = \{S \in \mathcal{F} : n \notin S\}$, where $a_n/b_n = \min_{i\in[n]} a_i/b_i$.

Membership testing

Submodularity testing amounts to solving

$$\frac{a_i}{b_i} + \frac{a_j}{b_j} \geq t_{ij} := \max_{S \in \mathcal{F}} \; h(S \cup \{i\}; a, b) + h(S \cup \{j\}; a, b)$$

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- Check if monotonicity holds over $\mathcal{F}_1 = \{S \in \mathcal{F} : n \in S\}$ and $\mathcal{F}_2 = \{S \in \mathcal{F} : n \notin S\}$
- Check if $t_{in} \leq a_i/b_i + a_n/b_n$ holds for all $i \in [n-1]$

Implications in computations

How can we benefit from submodularity?

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Submodular function maximization

Proposition (Nemhauser et al. (1978))

When the feasible region is given by a cardinality constraint,the greedy algorithm produces a solution with $(1 - e^{-1})$ *approx factor for* $\max_{S \in \mathcal{F}} f(S)$.

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Decomposition and cutting plane methods

$$h(x) = \frac{a'x}{1 + b'x} = \left(h(x) + \alpha \frac{b'x}{1 + b'x} \right) - \left(\alpha \frac{b'x}{1 + b'x} \right).$$

- epigraph ⇔ Lovász extension
- hypograph ⇒ valid inequalities

Computational experiment

Atamtürk and Narayanan (2021) consider

$$\min \left\{ \frac{a'x}{1 + b'x} - \Omega s'x : x \in \{0, 1\}^n \right\}$$

Benchmark:

Computational experiment

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Benchmark:

- Branch and Bound (B&B)

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Benchmark:

- Branch and Bound (B&B)
- B&B + cuts from submodular-supermodular decomposition

Computational results

λ	0.00	0.2	0.60	0.8	1.0
Gap(%)	1326.80	856.80	347.10	178.70	44.00
Cgap(%)	**90.80**	**61.50**	**21.50**	**9.60**	**0.00**
Time(s)	83.30	117.20	261.40	84.50	40.80
Ctime(s)	**44.10**	**88.90**	**71.10**	**12.30**	**0.00**
#Nodes	3.1E+04	3.6E+04	5.7E+04	3.2E+04	2.4E+04
#Cnodes	**1.6E+04**	**2.1E+04**	**2.2E+04**	**9.7E+03**	**0.0**
#Cuts	27.60	27.80	23.20	23.40	22.00

All computations are done with Gurobi version 9.0 on a Xeon workstation

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Thank You!

Reference

Atamtürk, A. and Narayanan, V. (2021). Submodular function minimization and polarity. *Mathematical Programming*, pages 1–11.

Benati, S. and Hansen, P. (2002). The maximum capture problem with random utilities: Problem formulation and algorithms. *European Journal of Operational Research*, 143(3):518–530.

Han, S., Gómez, A., and Prokopyev, O. A. (2020). Fractional 0-1 programming and submodularity. *arXiv preprint arXiv:2012.07235*.

Hansen, P., De Aragão, M. V. P., and Ribeiro, C. C. (1991). Hyperbolic 0–1 programming and query optimization in information retrieval. *Mathematical Programming*, 52(1-3):255–263.

Megiddo, N. et al. (1979). Combinatorial optimization with rational objective functions. *Mathematics of Operations Research*, 4(4):414–424.

Méndez-Díaz, I., Miranda-Bront, J. J., Vulcano, G., and Zabala, P. (2014). A branch-and-cut algorithm for the latent-class logit assortment problem. *Discrete Applied Mathematics*, 164:246–263.

Nemhauser, G. L., Wolsey, L. A., and Fisher, M. L. (1978). An analysis of approximations for maximizing submodular set functions—I. *Mathematical Programming*, 14(1):265–294.

Rusmevichientong, P., Shmoys, D., Tong, C., and Topaloglu, H. (2014). Assortment optimization under the multinomial logit model with random choice parameters. *Production and Operations Management*, 23(11):2023–2039.

Tawarmalani, M., Ahmed, S., and Sahinidis, N. V. (2002). Global optimization of 0-1 hyperbolic programs. *Journal of Global Optimization*, 24(4):385–416.

Ursulenko, O., Butenko, S., and Prokopyev, O. A. (2013). A global optimization algorithm for solving the minimum multiple ratio spanning tree problem. *Journal of Global Optimization*, 56(3):1029–1043.