Strongly polynomial algorithm for box-constrained quadratic programs with H_0-matrix

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March, 2022

Collaborator

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Motivation

Liu et al. (2021) investigate the following MIQP

$$\min_{x,z} a^\top z + c^\top x + \frac{1}{2} x^\top Q x$$
$$\text{s.t. } \mathbb{R}^n \ni x \perp z \in \{0,1\}^n,$$

where matrix Q is sparse positive definite and $x \perp z$ represents $x_i z_i = 0 \; \forall \, i$.

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in strongly linear time $\mathcal{O}(n)$.

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Question: What QPs are strongly polynomially solvable? Constraints?

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Consider a convex quadratic program

$$\min_{\ell \leq x \leq u} \; q^\top x + \frac{1}{2} x^\top M x, \tag{QP}$$

where $M \in \mathbb{S}^n_+$ is symmetric and positive semidefinite (PSD).

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Goal: Solve (QP) in strongly polynomial time under the assumption that M is an H_0-matrix.

H_0-matrices

Comparison matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \dots \\ m_{12} & m_{22} & m_{23} & \dots \\ m_{13} & m_{23} & m_{33} & \dots \\ \vdots & \vdots & \vdots & m_{nn} \end{bmatrix}$$

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H_0-matrix[1]: A matrix M is called an H_0-matrix if $\bar{M} \succeq 0$.

[1] There are at least 50 equivalent conditions under which M is an H-matrix!

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- M is an H_0-matrix $\Rightarrow$ $M \succeq 0$.
- $M \succeq 0$ is tridiagonal $\Rightarrow$ M is an H_0-matrix.

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H_0-matrices

Notation. Given a matrix $M \geq 0$ and $\alpha, \beta \subseteq [n] := \{1, 2, \ldots, n\}$

- submatrix $M_{\alpha\beta} := (m_{ij})_{i\in\alpha, j\in\beta}$
- the Schur complement of $M_{\alpha\alpha} \succ 0$: $(M/M_{\alpha\alpha}) := M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}}$, where $\bar{\alpha} = [n] \backslash \alpha$.

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Proposition (Operations preserving H_0-property)

Suppose M and N are two H_0-matrices. The following matrices are H_0-matrices.

- *Principle submatrix:* $M_{\alpha\alpha}$ *of* M, $\alpha \subseteq [n]$
- *Schur complement:* $(M/M_{\alpha\alpha})$
- *Nonnegative combination:* $\lambda_1 M + \lambda_2 N$ *where* $\lambda_1, \lambda_2 \geq 0$

H_0-matrices

Irreducibility

- M is *reducible* $\overset{\text{def}}{\iff}$ $M = \begin{bmatrix} M_{\alpha\alpha} & 0 \\ 0 & M_{\beta\beta} \end{bmatrix}$ (up to permutation), $\alpha, \beta \subset [n]$
- M is *irreducible* $\overset{\text{def}}{\iff}$ M is not reducible

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- Recognizing all irreducible blocks of M can be accomplished in $\mathcal{O}(n^2)$ complexity

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Proposition

If M is an irreducible H_0-matrix, then $\text{Rank}(M) \geq n - 1$.

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Key observation: if M is reducible,

$$\min_{\ell \leq x \leq u} q^\top x + \frac{1}{2} x^\top M x = \min_{\ell_\alpha \leq x_\alpha \leq u_\alpha} q_\alpha^\top x_\alpha + \frac{1}{2} x_\alpha^\top M_{\alpha\alpha} x_\alpha + \min_{\ell_\beta \leq x_\beta \leq u_\beta} q_\beta^\top x_\beta + \frac{1}{2} x_\beta^\top M_{\beta\beta} x_\beta$$

is separable.

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is separable. $\Rightarrow$ assume M is irreducible.

Parametric principal pivoting algorithm (PPPA)

WLOG, assume M is an irreducible H_0-matrix, $\ell = 0$ and $u \in \mathbb{R}^n$. Instead of directly solving

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PPPA traces the solution path of the parametric problem

$$\min_{0 \le x \le u} (q + \tau p)^\top x + \frac{1}{2} x^\top M x,$$

where $p \geq 0$ and $\exists \tau_0 > 0$ s.t. $q + \tau_0 p \geq 0$.

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Algorithm

- Initialization. $\alpha = \gamma = \emptyset, \beta = [n]$ and $\tau = \tau_0$ $\Rightarrow x^* = 0$

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Algorithm

- Initialization. $\alpha = \gamma = \varnothing, \beta = [n]$ and $\tau = \tau_0$ $\quad \Rightarrow x^* = 0$
- General iteration. Decrease τ until $\tau = 0$. At each break point, implement diagonal or 2×2 pivots and update index sets α, β, γ

$\tau = 0 \Rightarrow x^*$ is optimal for the original QP

Pivoting analysis

KKT/LCP

$$0 \leq w = q + \tau p + Mx + \lambda \perp x \geq 0$$
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$\Uparrow$

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Non-basic variables

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- H_0-property $\Rightarrow \bar{p}_\alpha \ge 0, \bar{p}_\gamma \ge 0, \oplus \ge 0.$

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- H_0-property $\Rightarrow$ $\bar{p}_\alpha \ge 0, \bar{p}_\gamma \ge 0, \oplus \ge 0$.
- Ratio test $\tau_{\text{new}} := \max\left\{\max_{i\in\alpha}\left\{-\frac{u_i + \bar{q}_i}{\bar{p}_i} : \bar{p}_i > 0\right\}, \max_{i\in\beta}\left\{-\frac{\bar{q}_i}{\bar{p}_i} : \bar{p}_i > 0\right\}, 0\right\}$

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  - $\bar{i} \in \alpha \to \alpha_{\text{new}} = \alpha \backslash \{\bar{i}\}, \gamma_{\text{new}} = \gamma \cup \{\bar{i}\}$

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Rearranging: $\alpha = \{i : 0 < x_i < u_i\}, \quad \beta = \{i : x_i = 0\}, \quad \gamma = \{i : x_i = u_i\}.$

$$\begin{array}{rcllllll}
0 \le x_\alpha &=& -\bar{q}_\alpha & \boxed{-\tau\bar{p}_\alpha} & + \Box w_\alpha + & \Box x_\beta + \Box s_\gamma + \Box\lambda_\alpha & & \perp w_\alpha \ge 0 \\
0 \le w_\beta &=& \bar{q}_\beta & -\tau\bar{p}_\beta & + \Box w_\alpha + & (M/M_{\alpha\alpha})_{\beta\beta} x_\beta + \Box s_\gamma + \Box\lambda_\alpha & + \lambda_\beta & \perp x_\beta \ge 0 \\
0 \le \lambda_\gamma &=& \Box & -\tau\bar{p}_\gamma & + \Box w_\alpha + & \Box x_\beta + \Box s_\gamma + \Box\lambda_\alpha & + w_\gamma & \perp s_\gamma \ge 0 \\
0 \le s_\alpha &=& \oplus & +\tau\bar{p}_\alpha & + \Box w_\alpha + & \Box x_\beta + \Box s_\gamma + \Box\lambda_\alpha & & \perp w_\alpha \ge 0 \\
0 \le s_\beta &=& u_\beta & & - & x_\beta & & \perp \lambda_\beta \ge 0 \\
0 \le x_\gamma &=& u_\gamma & & & - \; s_\gamma & & \perp w_\gamma \ge 0
\end{array}$$

- H_0-property $\Rightarrow \bar{p}_\alpha \ge 0, \bar{p}_\gamma \ge 0, \oplus \ge 0.$
- Ratio test $\tau_{\text{new}} := \max\left\{ \max_{i\in\alpha}\left\{ -\frac{u_i + \bar{q}_i}{\bar{p}_i} : \bar{p}_i > 0 \right\}, \max_{i\in\beta}\left\{ -\frac{\bar{q}_i}{\bar{p}_i} : \bar{p}_i > 0 \right\}, 0 \right\}$
  - $\bar{i} \in \alpha \to \alpha_{\text{new}} = \alpha \backslash \{\bar{i}\}, \gamma_{\text{new}} = \gamma \cup \{\bar{i}\}$
  - $\bar{i} \in \beta \to \begin{cases} (M/M_{\alpha\alpha})_{\bar{i}\bar{i}} \ne 0 \to \alpha_{\text{new}} = \alpha \cup \{\bar{i}\}, \beta_{\text{new}} = \beta \backslash \{\bar{i}\} \\ \end{cases}$

Pivoting analysis

KKT/LCP
$$0 \le w = q + \tau p + Mx + \lambda \perp x \ge 0$$
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0 \le \lambda_\gamma & = & \Box & - \tau \bar{p}_\gamma & + \Box w_\alpha & + & \Box x_\beta & + \Box s_\gamma & + \Box \lambda_\alpha & + w_\gamma & \perp s_\gamma \ge 0 \\
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Pivoting analysis

Note $\alpha = \{i : 0 < x_i < u_i\}, \quad \beta = \{i : x_i = 0\}, \quad \gamma = \{i : x_i = u_i\}$

Type of pivots

(1) $\alpha_{\text{new}} = \alpha \backslash \{\bar{i}\},\ \gamma_{\text{new}} = \gamma \cup \{\bar{i}\}$

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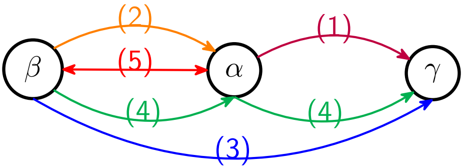

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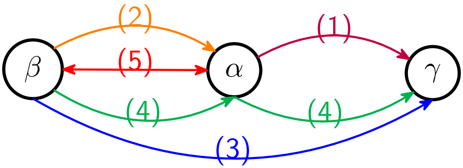
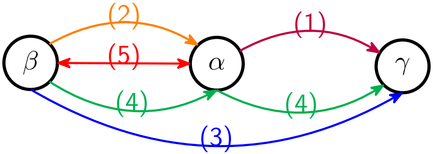


Theorem (Pang and Han (2021))

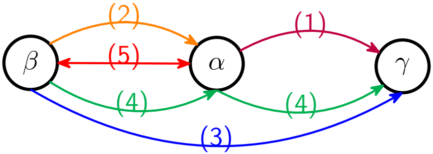
The streamlined PPPA terminates in at most $2n + 2$ pivots. Consequently, the overall complexity of the PPPA is of order $\mathcal{O}(n^3)$.

Iteration complexity of PPPA



Proof.

Iteration complexity of PPPA



Proof.

(a) No element can escape from γ

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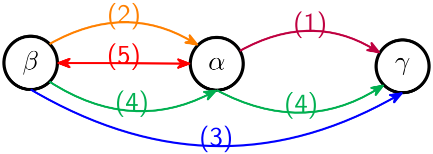

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(b) Except (1), all pivots lead to one element departing from β

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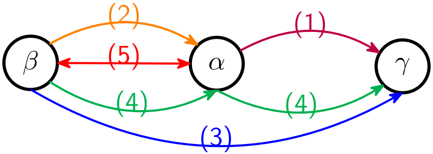

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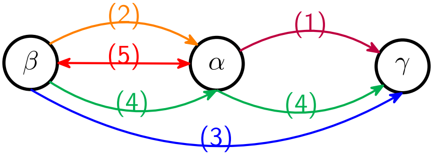
(c) There is only one backward arc due to (5)

Iteration complexity of PPPA



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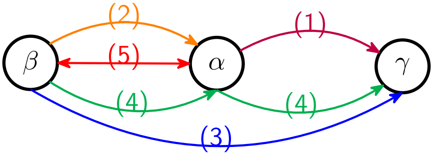

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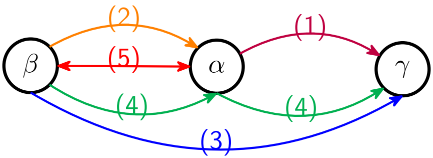

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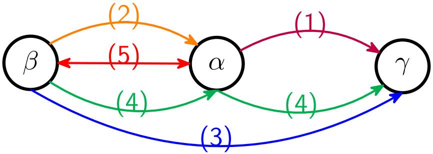

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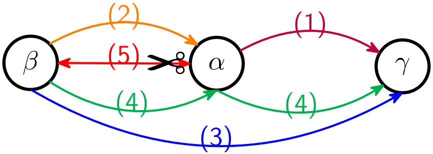

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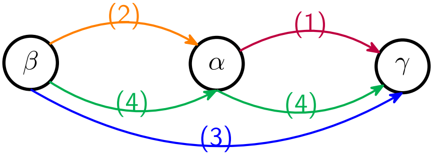

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\Rightarrow\ & \text{next pivot must be (1)} \Rightarrow \gamma_{\text{newnew}} \neq \varnothing \Rightarrow \text{No more } 2 \times 2 \text{ pivots!} \\
\Rightarrow\ & \text{Acyclic Graph}
\end{aligned}
$$

More discussion

Question: Existence of p such that $q + \tau_0 p \geq 0$? How to find it?

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Tridiagonal case

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Extension?

- k-weakly quasi-diagonally dominant matrices

Take home message

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Our paper is available at: `https://arxiv.org/abs/2112.03886`

Thank You!

Reference

Liu, P., Fattahi, S., Gomez, A., and Kucukyavuz, S. (2021). A graph-based decomposition method for convex quadratic optimization with indicators. *www.optimization-online.org/DB_HTML/2021/10/8645.html*.

Pang, J.-S. and Han, S. (2021). Some strongly polynomially solvable convex quadratic programs with bounded variables. *https://arxiv.org/abs/2112.03886*.